# Engineering Notes

*ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).*

## Modeling of Certain Strapdown Heading-Sensitive Errors in INS Error Models

Itzhack Y. Bar-Itzhack*
*Technion—Israel Institute of Technology, Haifa, Israel*

### Nomenclature

| | |
|---|---|
| $\bar A$ | = general vector |
| b | = body coordinate system |
| $bc$ | = body computed coordinate system |
| $c$ | = computed local level north pointing coordinate system |
| $D$ | = down axis |
| $D_c$ | = down axis of the $c$ coordinate system |
| $D_L^b$ | = transformation matrix from body to local level coordinate system |
| $E$ | = east axis |
| $F,F'$ | = dynamics matrices |
| $\bar f$ | = specific force vector |
| $\bar g$ | = gravity vector |
| $g$ | = magnitude of $\bar g$ |
| $\Delta\bar g$ | = error in computed $\bar g$ due to error in assumed position |
| $\delta\bar g$ | = gravity deflection and anomaly vector |
| $I$ | = identity matrix |
| $L$ | = local level north pointing coordinate system[2] |
| $N$ | = north axis |
| $p$ | = platform coordinate system[2] |
| $q$ | = general coordinate system |
| $\Delta\bar R$ | = position error vector |
| $t$ | = true coordinate system[2] |
| $T$ | = transpose |
| $\Delta\bar V$ | = INS-computed velocity error vector |
| $\boldsymbol{w}$ | = white noise column vector |
| $\boldsymbol{x}$ | = INS error state vector |
| $\bar\epsilon$ | = gyro drift rate vector |
| $\delta\bar\theta$ | = vector angle by which a rotation of the $t$ coordinate system ends at the $c$ coordinate system[2] |
| $\bar\rho$ | = angular rotation rate vector of the $L$ with respect to the $c$ coordinate system[2] |
| $\bar\phi$ | = vector angle by which a rotation of the $t$ coordinate system ends at the $p$ coordinate system[2] |
| $\bar\psi$ | = vector angle by which a rotation of the $c$ coordinate system ends at the $p$ coordinate system[2] |
| $\bar\Omega$ | = Earth rate vector |
| $[\boldsymbol{\phi}_t\times]$ | = vector product matrix of $\boldsymbol{\phi}_t$ |
| $\bar\omega$ | = angular rotation rate vector of the $t$ coordinate system with respect to an inertial coordinate system[2] |
| $\bar\nabla$ | = accelerometer error vector |
| $\underline{A}_q$ | = column vector whose components are the components of the general vector $\bar A$ when resolved in coordinate system $q$ |
| $[\boldsymbol{\phi}_t\times]$ | = vector product matrix of $\boldsymbol{\phi}_t$ |
| $\overset{q}{\overline{A}}$ | = rate of change of vector $\bar A$ as seen by an observer in coordinate system $q$ |

### Introduction

SELF-ALIGNMENT of a gimbaled inertial navigation system (INS) results in a platform tilt which cancels the effect of the level accelerometer biases. The same cancellation takes place in strapdown INS too; however, unlike gimbaled INS, in a strapdown system, this cancellation is perturbed once the INS changes heading. This Note shows that the standard strapdown INS error model does describe this heading-sensitive phenomenon and that the transformation matrix from the body to the reference coordinate system plays a major role in the modeling of this phenomenon.

The differential equations which describe the error propagation in INS are divided into equations which describe the propagation of the translatory errors and equations which describe the propagation of the attitude errors.

The propagation of the translatory errors, namely position and velocity errors, is classically described by either[1,2]

$$\overset{tt}{\overline{\Delta\bar R}}+2\bar\omega\times\overset{t}{\overline{\Delta\bar R}}+\overset{t}{\overline{\bar\omega}}\times\Delta\bar R+\bar\omega\times(\bar\omega\times\Delta\bar R)-\bar\Omega\times(\bar\Omega\times\Delta\bar R)$$

$$=\bar\nabla-\bar\psi\times\bar f+\Delta\bar g+\delta\bar g \tag{1a}$$

$$\Delta\bar V=\overset{t}{\overline{\Delta\bar R}}+\bar\rho\times\Delta\bar R \tag{1b}$$

or[3,4]

$$\overset{t}{\overline{\Delta\bar V}}+(\bar\Omega+\bar\omega)\times\Delta\bar V=\bar\nabla-\bar\psi\times\bar f+\Delta\bar g+\delta\bar g \tag{2a}$$

$$\overline{\Delta\bar R}+\bar\rho\times\Delta\bar R=\Delta\bar V \tag{2b}$$

where Eqs. (1a) and (2a) are, of course, identical.

The propagation of the attitude errors is given by[1,2]

$$\overset{t}{\bar\psi}+\bar\omega\times\bar\psi=\pm\bar\epsilon \tag{3}$$

(The plus sign in front of $\bar\epsilon$ is chosen when the INS is a gimbaled one and the minus sign is chosen for a strapdown INS[5,6]). Using the following realtionship between $\delta\bar\theta$ and $\bar\psi$

$$\bar\psi=\bar\phi-\delta\bar\theta \tag{4}$$

Equation (3) can be transformed into the following equation which can be used instead of Eq. (3)

$$\overset{t}{\bar\phi}+\bar\omega\times\bar\phi=\pm\bar\epsilon+\overset{t}{\overline{\delta\bar\theta}}+\bar\omega\times\delta\bar\theta \tag{5}$$

As seen from Eqs. (1a) and (2a), the attitude error equations have to be solved too in order to solve the translatory error equations.

Received Nov. 2, 1983. 

*Professor, Aeronautical Engineering Department.

Usually the INS error equations are transformed into the state space where they are used in Kalman filtering or in analyzing system performance utilizing covariance propagation methods. In the state space, the INS error model is presented in the following form

$$\dot{x} = Fx + w \tag{6}$$

where $x$ contains the INS position, velocity and attitude errors, as well as states describing the accelerometer and gyro error models, and where $w$ is a white noise vector. Assume, for example, that the model given in Eqs. (2), (4), and (5) is transformed into the state space and further assume that the accelerometer models, as well as the gyro models, consist of a random constant, a white noise and a random walk component. Then, $x$ is given by

$$\underline{x}^T = [\Delta \underline{R}_t^T, \Delta \underline{V}_t^T, \underline{\phi}_t^T, \underline{\nabla}_t^T, \underline{\epsilon}_t^T] \tag{7}$$

where an underbarred quantity denotes a column matrix whose elements are the elements of the pertinent vector when resolved in the reference $t$ coordinate system. The matrix $F$ is of the following form

$$F = \left[\begin{array}{c|c|c} F' & \begin{array}{c} \\ I \\ \\ \end{array} & \begin{array}{c} \\ \\ I \end{array} \\ \hline & & \\ \hline & & \end{array}\right] \tag{8}$$

where $F'$ describes the dynamic relations between $\Delta R_t$, $\Delta V_t$ and $\phi_t$. These relations are expressed in Eqs. (2), (4), and (5). The specific form of $F'$ is obtained when resolving the vectorial expressions of those equations in the reference, i.e., $t$, coordinate system.

The state space representation of the error model of strapdown INS is almost identical to the preceding model of gimbaled INS with the only differences[7] being that the $3 \times 3$ submatrices $I$ in the matrix $F$, given in Eq. (8), are replaced by $D_t^b$ ($D_t^b$ is the transformation matrix which transforms vectors from the body $b$ to the reference coordinate system) and that the state vector of Eq. (7) is now

$$\underline{x}^T = [\Delta \underline{R}_t^T, \Delta \underline{V}_t^T, \underline{\phi}_t^T, \underline{\nabla}_b^T, \underline{\epsilon}_b^T] \tag{9}$$

It is well known that self-alignment of a gimbaled INS produces platform tilts which, through $\bar{g}$, practically cancel the effect of the level accelerometer biases. This situation remains unchanged even when the vehicle changes its heading, since the platform attitude remains unchanged. In strapdown INS also, the effect of accelerometer bias compensation by the tilt errors after self-alignment, is well known. (Note that in a strapdown INS the physical tilts of the gimbaled system are replaced by mathematical tilts.) However, in strapdown INS, this bias compensation is destroyed once the vehicle changes its heading,[8,9] the worst case bing a heading change of 180 deg.[8] Two questions are then asked:

1) Does the present error propagation model describe this phenomenon?

2) If so, where in the model is it described?

## Analysis

Since the only difference between the translatory error models of the gimbaled and the strapdown INS is in the introduction of $D_t^b$ in the strapdown model, it could be justifiably concluded that if the model accounts for the heading sensitivity, then it is $D_t^b$ which is involved in the modeling of this error. To show specifically how this takes place, consider the following analysis:

From the strapdown INS error model outlined here it can be shown that

$$\Delta \dot{\underline{V}}_t = -[\underline{\Omega}_t \times]\Delta \underline{V}_t + [\underline{\phi}_t \times]\underline{g}_t + D_t^b \underline{\nabla}_b \tag{10}$$

Let the reference, $t$, coordinate system be the local level north-pointing $L$ coordinate system and assume that before self-alignment commences, the body north axis happens to coincide with the north axis of the $L$ frame and, similarly, the body east axis coincides with the $L$ frame east axis (i.e., they correspondingly point in the local north and east directions.) Then, at the end of the self-alignment stage, when $\Delta \dot{\underline{V}}_L = \Delta \underline{V}_L = \underline{0}$, the east component of Eq. (10) satisfies

$$-\phi_{NL}(0)g + \left\{ \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} \nabla_{Nb} \\ \nabla_{Eb} \\ \nabla_{Db} \end{bmatrix} \right\}_{EL} = 0 \tag{11}$$

which yields

$$\phi_{NL}(0) = \frac{\nabla_{Eb}}{g} \tag{12}$$

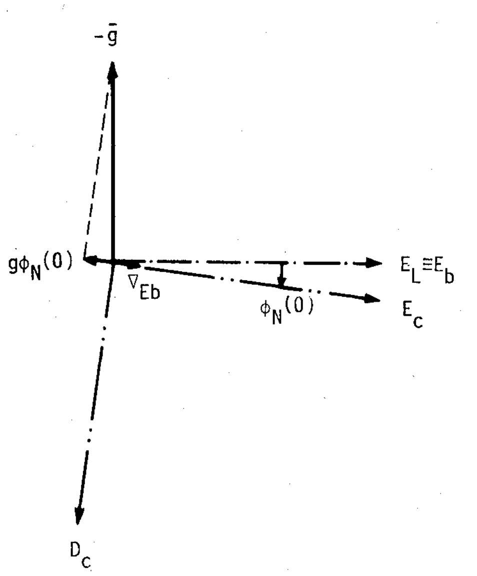


**Fig. 1 Relative orientation between local level north-pointing coordinate system and computed strapdown coordinate system at the end of self-alignment.**

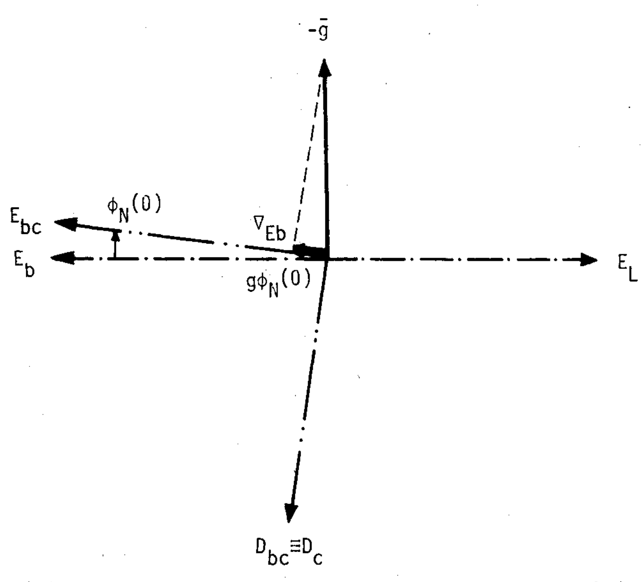


**Fig. 2 Relative orientation between local level north-pointing coordinate system and computed strapdown coordinate system after a 180 deg azimuth change.**

Hypothesizing that the INS undergoes a 180 deg heading change, then

$$D_L^b = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \tag{13}$$

thus, using Eqs. (12) and (13) in Eq. (10) yields the following east axis equation

$$\Delta \dot{V}_{EL} = -\frac{\nabla_{Eb}}{g} g + \left( \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} \nabla_{Nb} \\ \nabla_{Eb} \\ \nabla_{Db} \end{bmatrix} \right)_{EL} \tag{14}$$

which can be written as

$$\Delta \dot{V}_{EL} = -2\nabla_{Eb} \tag{15a}$$

Using Eq. (12), this can be equivalently written as

$$\Delta \dot{V}_{EL} = -2g\phi_{NL}(0) \tag{15b}$$

Equations (15a) and (15b) express the heading sensitivity phenomenon in strapdown INS.

The preceding analysis can be illustrated graphically as follows. Figure 1 presents the goemetry at the end of the self-alignment stage while Fig. 2 presents the geometry after a 180 deg heading change. In this special example, the body axes $b$ are chosen to coincide with the local level north-pointing axes $L$, thus $E_b \equiv E_L$; however, the *computed* local level north-pointing east, $E_c$, and down, $D_c$, axes are tilted by the misalignment angle $\phi_N(0)$ about the north axis. Therefore, when the *computed* transformation matrix, which transforms vector from the $b$ to the $L$ coordinate system, is used to transform the vector of accelerometer biases from the $b$ to the $L$ system, the result is the vector $\nabla_{Eb}$ along the computed east axis, $E_c$. As is well known, in the self-alignment stage the tilt angle is determined computationally in such a way as to satisfy Eq. (12). Indeed, the geometry of Fig. 1 expresses the equality in magnitude of the two opposing vectors $g\phi_N(0)$ and $\nabla_{Eb}$ such that

$$-g\phi_N(0) + \nabla_{Eb} = 0 \tag{16}$$

from which Eq. (12) stems. (Note that the sensed gravity vector is interpreted by the accelerometer as an acceleration in a direction opposite to the gravity vecotr. For this reason $\bar{g}$ is plotted upwards). When the inertial measuring unit of the strapdown INS is now rotated in azimuth (about the down $D$ axis, not shown in the figures) at a 180 deg angle, this rotation is picked up by the strapdown down gyro and fed into the strapdown computer which computes a new transformation matrix. Due to the initial misalignment error, the newly computed transformation matrix implies that the body east axis, $E_{bc}$, differs from the true body east axis, $E_b$, by the same initial misalignment error $\phi_N(0)$. It also implies that the accelerometer which points in the $E_b$ direction measures along the $E_{bc}$ axis the quantity $g\phi_N(0)$ in addition to the accelerometer bias which is equal in magnitude to $\nabla_{Eb}$. That is, it assumes that $f_{Eb}$, the specific force measured in the body east direction, is

$$f_{Eb} = \nabla_{Eb} + g\phi_N(0) \tag{17}$$

and then since $D_L^{bc}$, the transformation matrix from the $bc$ to the $L$ coordinate system, is as follows

$$D_L^{bc} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -\phi_N(0) \\ 0 & -\phi_N(0) & 1 \end{bmatrix} \tag{18}$$

then when the vector of specific forces whose east value is $f_{Eb}$ of Eq. (17) is transformed by $D_L^b$ into the $L$ system the result to the first order is

$$\Delta \dot{V}_{EL} = -[\nabla_{Eb} + g\phi_N(0)] \tag{19}$$

or by using Eq. (16), Eqs. (15) are obtainable.

Although a special case was used to demonstrate our argument, one can easily use any initial orientation of the body axes (i.e., not necessarily a coincidence with the $L$ system) and any azimuth change (i.e., not necessarily 180 deg change). The argumentation will then be more involved, but the conclusions will be identical.

## References

[1] Pitman, G.R. Jr., ed., *Inertial Guidance,* John Wiley & Sons, New York, 1962.

[2] Leondes, C.T., ed., *Guidance and Control of Aerospace Vehicles,* McGraw-Hill, New York, 1963.

[3] Benson, D.O. Jr., "A Comparison of Two Approaches to Pure-Inertial and Doppler-Inertial Error Analysis," *IEEE Transactions on Aerospace and Electronic Systems,* Vol. AES-11, July, 1975, pp. 447-455.

[4] D'Appolito, J.A., "The Evaluation of Kalman Filter Designs for Multisensor Integrated Navigation Systems," AFAL-TR-70-217 (AD881 286), The Analytic Sciences Corporation, Reading, Mass., Jan. 1971.

[5] Weinreb, A. and Bar-Itzhack, I.Y., "The Psi-Angle Error Equation in Strapdown Inertial Navigation Systems," *IEEE Transactions on Aerospace and Electronic Systems,* Vol. AES-14, May 1978, pp. 539-542.

[6] Weinreb, A. and Bar-Itzhack, I.Y., "Authors reply," to D.O. Benson, Jr., "Comments on 'The Psi-Angle Error Equation in Strapdown Inertial Navigation Systems,'" *IEEE Transactions on Aerospace and Electronic Systems,* Vol. AES-15, Jan. 1979, pp. 168-170.

[7] Bar-Itzhack, I.Y., Serfaty, D. and Vitek, Y., "Doppler-Aided Low-Accuracy Strapdown Inertial Navigation System," *Journal of Guidance and Control,* Vol. 5, May-June 1982, p. 236-242.

[8] Stambaugh, J.S., "Propagation and System Accuracy Impact of Major Sensor Errors on a Strapdown Aircraft Navigator," *IEEE Transactions of Aerospace and Electronic Systems,* Vol. AES-9, Nov. 1973, pp. 838-246.

[9] Schmidt, G.T., "Strapdown Inertial Navigation Systems—Theory Applications Introduction and Overview," in *Strap-Down Inertial Systems,"* NATO AGARD-LS-95, AD/A-055 778, May 1978.

# The General Class of Optimal Proportional Navigation

Wolfgang Pfleghaar*
*Messerschmitt-Bölkow-Blohm,*
*Munich, Federal Republic of Germany*

## Introduction

PROPORTIONAL navigation (PN) is a well-known homing intercept guidance law whose performance has been studied intensively.[1,2] Theoretical and practical experience with proportional navigation has shown that the navigation ratio $N'$ is one of the most important design parameters for PN guidance schemes to date. The ratio $N'$ is, in practice, held fixed with acceptable values ($3 < N' < 5$) determined by noise, radome, and target maneuver considerations.[2] With the appli-

Received Feb. 9, 1982; revision received Nov. 29, 1983. 

*Member Technical Staff, Dynamics Division.